Homework 10 Solution

1. Sec. 6.1 Q21

21. Let A be an $n \times n$ matrix. Define

$$A_1 = \frac{1}{2}(A + A^*) \quad \text{and} \quad A_2 = \frac{1}{2i}(A - A^*).$$

(a) Prove that $A_1^* = A_1$, $A_2^* = A_2$, and $A = A_1 + iA_2$. Would it be reasonable to define A_1 and A_2 to be the real and imaginary parts, respectively, of the matrix A?

(b) Let A be an $n \times n$ matrix. Prove that the representation in (a) is unique. That is, prove that if $A = B_1 + iB_2$, where $B_1^* = B_1$ and $B_2^* = B_2$, then $B_1 = A_1$ and $B_2 = A_2$.

(a)

- $$(A_1)^* = \left(\frac{1}{2}(A + A^*)\right)^* = \frac{1}{2}(A^* + A^{**})$$
  $$= \frac{1}{2}(A^* + A) = A_1$$

- $$(A_2)^* = \left(\frac{1}{2i}(A - A^*)\right)^* = -\frac{1}{2i}(A^* - A^{**})$$
  $$= \frac{1}{2i}(A - A^*) = A_2$$

- $$A_1 + iA_2 = \frac{1}{2}(A + A^*) + i \cdot \frac{1}{2i}(A - A^*)$$
  $$= \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*)$$
  $$= A$$

- It's reasonable.

  $A = A_1 + iA_2$ where $A_1^* = A_1$, $A_2^* = A_2$

  $A^* = (A_1 + iA_2)^* = A_1^* - iA_2^* = A_1 - iA_2$

  Analogous to the def of Complex number

  $x = a + ib$ where $a, b \in \mathbb{R}$

  $\bar{x} = \overline{a + ib} = \bar{a} - i\bar{b} = a - ib$

(b)

if $A = A_1 + i A_2$ where $A_1^* = A_1$ $A_2^* = A_2$

$A = B_1 + i B_2$ where $B_1^* = B_1$ $B_2^* = B_2$

then $\begin{cases} A^* = A_1 - i A_2 \\ A^* = B_1 - i B_2 \end{cases}$

$\begin{cases} 2A_1 = A + A^* = 2B_1 \\ 2i A_2 = A - A^* = 2i B_2 \end{cases}$

i.e. $A_1 = B_1$ $A_2 = B_2$

Thus it's more reasonable to define A_1 and A_2 to be the real and imaginary parts of A.

2. Sec. 6.2 Q14

14. Let W_1 and W_2 be subspaces of a finite-dimensional inner product space. Prove that $(W_1+W_2)^\perp = W_1^\perp \cap W_2^\perp$ and $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$. (See the definition of the sum of subsets of a vector space on page 22.) *Hint for the second equation:* Apply Exercise 13(c) to the first equation.

① we prove $(W_1+W_2)^\perp = W_1^\perp \cap W_2^\perp$

- $\forall x \in (W_1+W_2)^\perp$, we have $\langle x, y\rangle = 0 \quad \forall y \in W_1+W_2$

  Then $\langle x, y\rangle = 0 \quad \forall y \in W_1 \subset W_1+W_2$ i.e. $x \in W_1^\perp$

  Similarly. $x \in W_2^\perp$. Thus $x \in W_1^\perp \cap W_2^\perp$

- $\forall x \in W_1^\perp \cap W_2^\perp$

  $x \in W_1^\perp$, so $\langle x, y\rangle = 0 \quad \forall y \in W_1$

  $x \in W_2^\perp$ so $\langle x, y\rangle = 0 \quad \forall y \in W_2$

  $\forall y \in W_1+W_2$. $\exists y_1 \in W_1$. $y_2 \in W_2$ st $y = y_1+y_2$

  $\langle x, y\rangle = \langle x, y_1\rangle + \langle x, y_2\rangle = 0$

  So $x \in (W_1+W_2)^\perp$

② we prove $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$

By ①, we have $(W_1^\perp + W_2^\perp)^\perp = (W_1^\perp)^\perp \cap (W_2^\perp)^\perp$

By exercise 13(c), $W_1 = (W_1^\perp)^\perp \quad W_2 = (W_2^\perp)^\perp$

$W_1^\perp + W_2^\perp = \left((W_1^\perp + W_2^\perp)^\perp\right)^\perp$

Therefore

$W_1^\perp + W_2^\perp = \left((W_1^\perp+W_2^\perp)^\perp\right)^\perp = \left((W_1^\perp)^\perp \cap (W_2^\perp)^\perp\right)^\perp = (W_1 \cap W_2)^\perp$

3. Sec. 6.2 Q23

23. Let V be the vector space defined in Example 5 of Section 1.2, the space of all sequences σ in F (where $F = R$ or $F = C$) such that $\sigma(n) \neq 0$ for only finitely many positive integers n. For $\sigma, \mu \in$ V, we define $\langle \sigma, \mu \rangle = \sum_{n=1}^{\infty} \sigma(n)\overline{\mu(n)}$. Since all but a finite number of terms of the series are zero, the series converges.

(a) Prove that $\langle \cdot, \cdot \rangle$ is an inner product on V, and hence V is an inner product space.

(b) For each positive integer n, let e_n be the sequence defined by $e_n(k) = \delta_{n,k}$, where $\delta_{n,k}$ is the Kronecker delta. Prove that $\{e_1, e_2, \ldots\}$ is an orthonormal basis for V.

(c) Let $\sigma_n = e_1 + e_n$ and W $=$ span$(\{\sigma_n : n \geq 2\}$.

(i) Prove that $e_1 \notin$ W, so W $\neq$ V.

(ii) Prove that W$^{\perp} = \{0\}$, and conclude that W $\neq$ (W$^{\perp}$)$^{\perp}$.

Thus the assumption in Exercise 13(c) that W is finite-dimensional is essential.

(a) easy to check with the def of inner product.

(b) $\langle e_i, e_j \rangle = \sum_{n=1}^{\infty} e_i(n) \cdot \overline{e_j(n)} = \sum_{n=1}^{\infty} \delta_{i,n} \cdot \overline{\delta_{j,n}} = \delta_{ij}$

So $\{e_i\}_1^{\infty}$ is orthonormal subset of V

$\forall \sigma \in V$. $S = \{n : \sigma(n) \neq 0\}$ is a finite set

Then $\sigma = \sum_{n \in S} \sigma(n) \cdot e_n \in \text{span}(\{e_i\}_1^{\infty})$

Therefore. $\{e_i\}_1^{\infty}$ is an orthonormal basis for V.

(c)

(i) if $e_1 \in W$, then $e_1 = a_2 \cdot \sigma_2 + \cdots + a_n \cdot \sigma_n$ for some $a_1 \cdots a_n \in F$

Then $(a_2 + \cdots + a_n - 1) e_1 + a_2 e_2 + \cdots + a_n e_n = 0$

Since $\{e_1 \cdots e_n\} \subset \{e_j\}_1^{\infty}$ is L.I.

we have $a_2 + \cdots + a_n - 1 = a_2 = \cdots = a_n = 0$ which is impossible.

So $e_1 \notin W$ and thus $W \neq V$

(ii)

$\forall v \in W^\perp$, we have $\langle v, \sigma_n \rangle = 0$ for $n \geq 2$

i.e. $\langle v, e_1 \rangle = -\langle v, e_n \rangle \quad \forall n \geq 2$

i.e. $v(1) = -v(n) \quad \forall n \geq 2$

If $v(1) \neq 0$, Then $v(n) \neq 0 \quad \forall n \geq 2$.

which implies $v \notin V$. contradiction!

Thus $v(1) = 0$ and $v(n) = -v(1) = 0 \quad \forall n \geq 2$

i.e. $v = 0 \in V$.

Thus $W^\perp = \{0\}$

we conclude that $W \neq V = \{0\}^\perp = (W^\perp)^\perp$

Therefore, that W is finite-dim is essential in exercise 13(c)

4. Sec. 6.3 Q9

9. Prove that if $\mathsf{V} = \mathsf{W} \oplus \mathsf{W}^{\perp}$ and T is the projection on W along $\mathsf{W}^{\perp}$, then $\mathsf{T} = \mathsf{T}^*$. *Hint:* Recall that $\mathsf{N}(\mathsf{T}) = \mathsf{W}^{\perp}$. (For definitions, see the exercises of Sections 1.3 and 2.1.)

$\forall v \in W,\quad \exists!\ w_1 \in W$ and $w_2 \in W^{\perp}$

s.t. $\quad v = w_1 + w_2$

And $\quad T(v) = w_1$

$\forall x, y \in V.\quad \exists!\ x_1, y_1 \in W$ and $x_2, y_2 \in W^{\perp}$

$$\text{s.t.} \begin{cases} x = x_1 + x_2, & T(x) = x_1 \\ y = y_1 + y_2, & T(y) = y_1 \end{cases}$$

$$\begin{aligned}
\langle T(x), y \rangle &= \langle x_1, y \rangle \\
&= \langle x_1, y_1 + y_2 \rangle \\
&= \langle x_1, y_1 \rangle + \langle \underset{\in W}{x_1}, \underset{\in W^{\perp}}{y_2} \rangle_{=0} \\
&= \langle x_1, y_1 \rangle \\
&= \langle x_1, y_1 \rangle + \langle \underset{\in W^{\perp}}{x_2}, \underset{\in W}{y_1} \rangle_{=0} \\
&= \langle x_1 + x_2, y_1 \rangle \\
&= \langle x, T(y) \rangle
\end{aligned}$$

Therefore $\quad T = T^*$

5. Sec. 6.3 Q14

14. Let V be an inner product space, and let $y, z \in \mathsf{V}$. Define $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ by $\mathsf{T}(x) = \langle x, y\rangle z$ for all $x \in \mathsf{V}$. First prove that T is linear. Then show that T^* exists, and find an explicit expression for it.

- T is linear.

$$\begin{aligned} T(cx_1 + x_2) &= \langle cx_1 + x_2, y\rangle \cdot z \\ &= (c\langle x_1, y\rangle + \langle x_2, y\rangle) \cdot z \\ &= c\cdot(\langle x_1, y\rangle \cdot z) + (\langle x_2, y\rangle \cdot z) \\ &= c \cdot T(x_1) + T(x_2) \end{aligned}$$

- $\forall\ x_1, x_2 \in V.$

$$\begin{aligned} \langle T(x_1), x_2\rangle &= \langle \langle x_1, y\rangle \cdot z, x_2\rangle \\ &= \langle x_1, y\rangle \cdot \langle z, x_2\rangle \\ &= \langle x_1, \overline{\langle z, x_2\rangle} \cdot y\rangle \\ &= \langle x_1, \langle x_2, z\rangle \cdot y\rangle \end{aligned}$$

$$\langle T(x_1), x_2\rangle = \langle x_1, T^*(x_2)\rangle$$

implies that $T^*(x) = \langle x, z\rangle \cdot y$.